CONTROL VOLUME APPROACH

Derivation of Continuity Equation

$$\frac{dM_{sys}}{dt} = \frac{d}{dt}\int_{CV} \rho dV + \int_{CS} \rho V \bullet dA$$

$$\frac{dB_{sys}}{dt} = \frac{d}{dt}\int_{cv} b\rho dQ + \sum_{cs} \dot{m}_{out} b_{out} - \sum_{cs} \dot{m}_{in} b_{in}$$

But $\frac{dM_{sys}}{dt} = 0$ $\quad M_{sys} = \text{Const}$

$$\frac{d}{dt}\int_{CV} \rho dV + \int_{CS} \rho V \bullet dA = 0$$

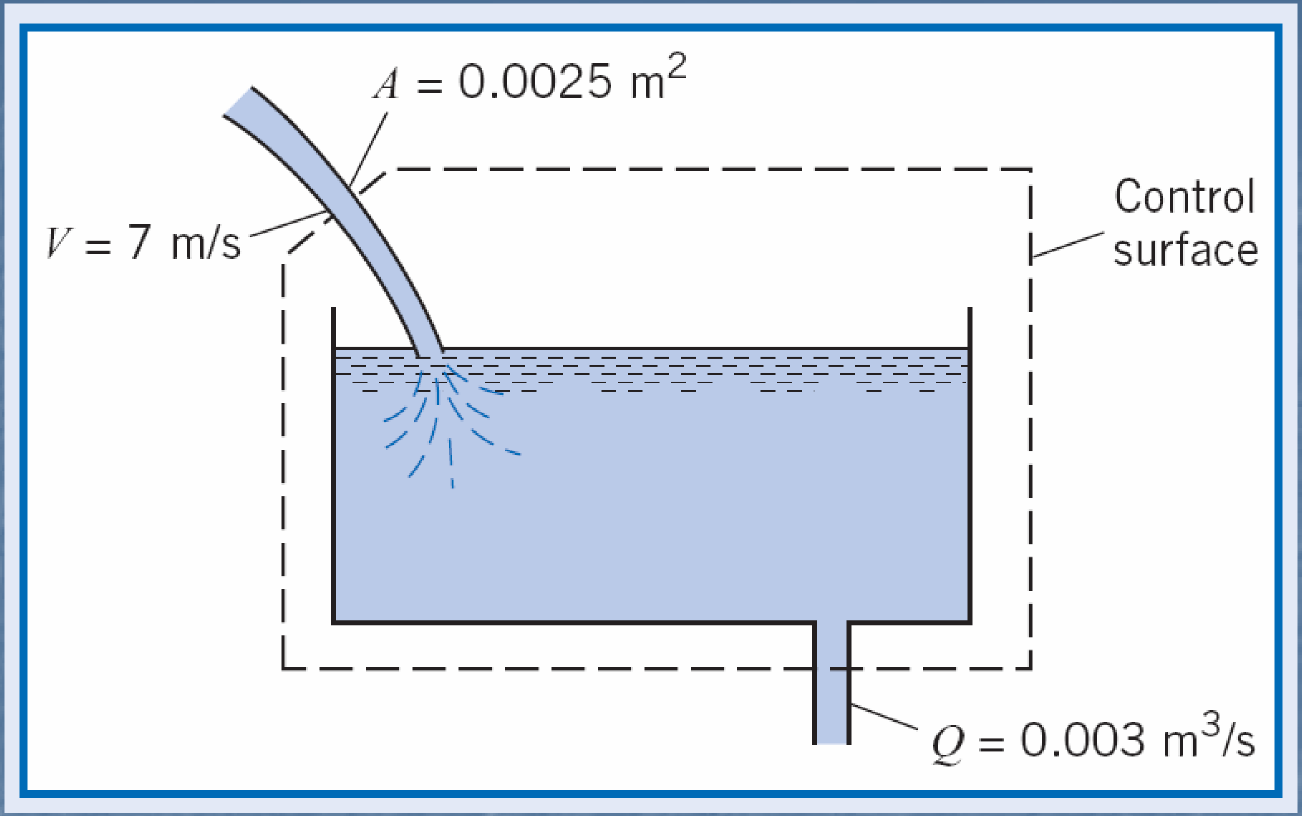
$$\frac{dM_{CV}}{dt} + \sum_{cs} \dot{m}_{out} - \sum_{cs} \dot{m}_{in} = 0 \quad \rightarrow \quad \frac{d}{dt} M_{cv} = \dot{m}_{in} - \dot{m}_{out} \qquad \frac{d}{dt}\left(\dot{Q}_{CV}\right) = \dot{Q}_{in} - \dot{Q}_{out}$$

For Steady Flow in a Pipe

$$\left(\frac{dM_{CV}}{dt}\right) = 0$$

$$\dot{m}_{in} = \dot{m}_{out}$$

Example 5.5 (p. 155)



Find the rate of water accumulating in or evacuating from the tank?

$$\left(\frac{dM_{CV}}{dt}\right) = ?$$

Solution By drawing a control surface to enclose the entire tank, we observe that there are two streams crossing the control surface, one entering and one leaving. Applying the continuity equation, we have

$$\frac{dM_{cv}}{dt} = \dot{m}_i - \dot{m}_o$$

The mass flow rate out is

$$\dot{m}_o = \rho Q = 1000\ \mathrm{kg/m^3} \times 0.003\ \mathrm{m^3/s}$$
$$= 3\ \mathrm{kg/s}$$

The mass flow rate in is

$$\dot{m}_i = \rho V A$$
$$= 1000\ \mathrm{kg/m^3} \times 7\ \mathrm{m/s} \times 0.0025\ \mathrm{m^2}$$
$$= 17.5\ \mathrm{kg/s}$$

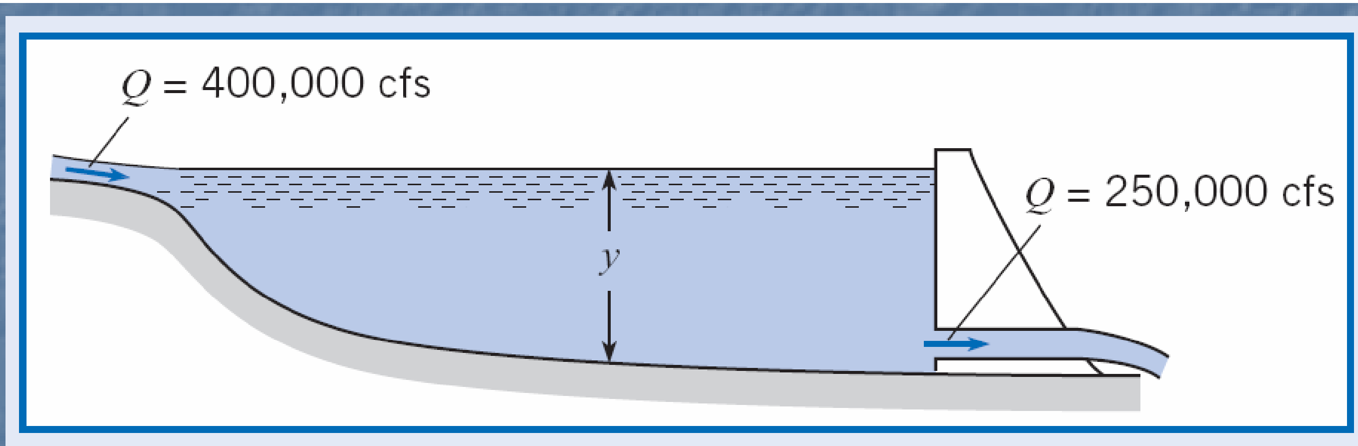
From the continuity equation,

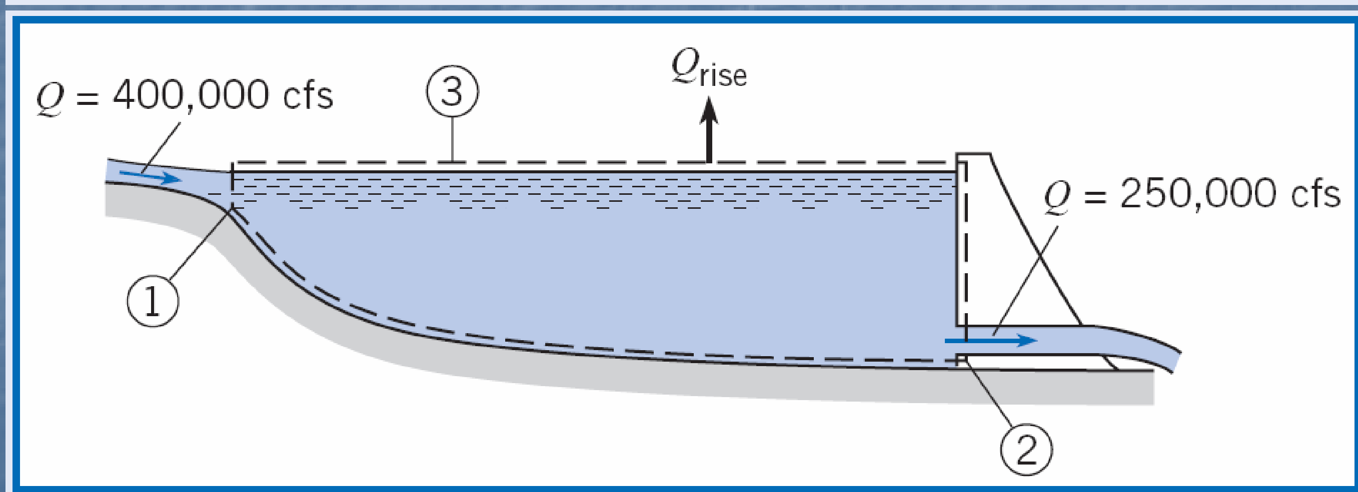
$$\frac{dM_{cv}}{dt} = 17.5 - 3$$
$$= 14.5\ \mathrm{kg/s} \quad \triangleleft$$

so the mass is accumulating in the tank at the rate of 14.5 kg/s.

Example 5.6b (p. 156)

The river discharges into the reservoir shown at a rate of 400,000 ft^3/s (cfs), and the outflow rate from the reservoir through the flow passages in the dam is 250,000 cfs. If the reservoir surface area is 40 mi^2, what is the rate of rise of water in the reservoir?





Find the rate of rise of water in the reservoir? (dh/dt)

$$\sum_{cs} \dot{m}_o - \sum_{cs} \dot{m}_i = 0$$

and because ρ is constant,

$$\sum_{cs} Q_o - \sum_{cs} Q_i = 0$$

Applied to this example, we have

$$Q_3 + Q_2 - Q_1 = 0$$

$$Q_{rise} + 250{,}000\ \text{ft}^3/\text{s} - 400{,}000\ \text{ft}^3/\text{s} = 0$$

$$Q_{rise} = 150{,}000\ \text{ft}^3/\text{s}$$

Q_{rise} is related to V_{rise} by $V_{rise} A_R$, where A_R is the area of the reservoir. Then we have

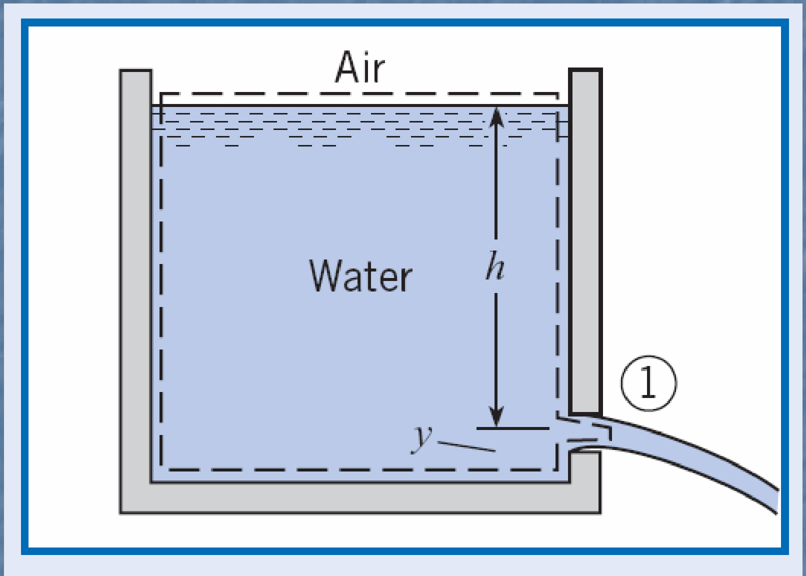
$$V_{rise} = \frac{150{,}000\ \text{ft}^3/\text{s}}{40\ \text{mi}^2 \times (5280)^2 \text{ft}^2/\text{mi}^2}$$

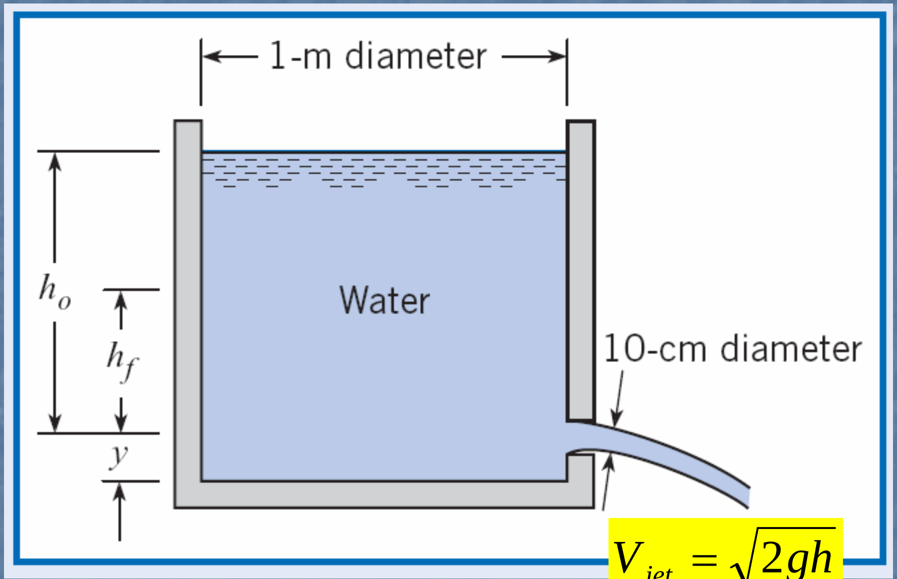
$$\text{Rate of rise} = 1.34 \times 10^{-4}\ \text{ft/s}$$

or

$$V_{rise} = 0.484\ \text{ft/hr}$$

Example 5.7a (p. 157)





$$V_{jet} = \sqrt{2gh}$$

How long will take (t=?) for the water to drop from h0= 2m to hf=0.5 ?

$$h_0 = 2m$$

$$h_f = 0.5m$$

$$\frac{dM_{CV}}{dt} = \sum_{cs} \dot{m}_{in} - \sum_{cs} \dot{m}_{out}$$

$$\sum_{cs} \dot{m}_{in} = 0$$

$$\frac{dM_{CV}}{dt} = -\sum_{CS} \dot{m}_{out}$$

$$-(\rho AV)_{out} = \frac{d}{dt}(\rho A_T)(h+y) \qquad \frac{dy}{dt} = 0$$

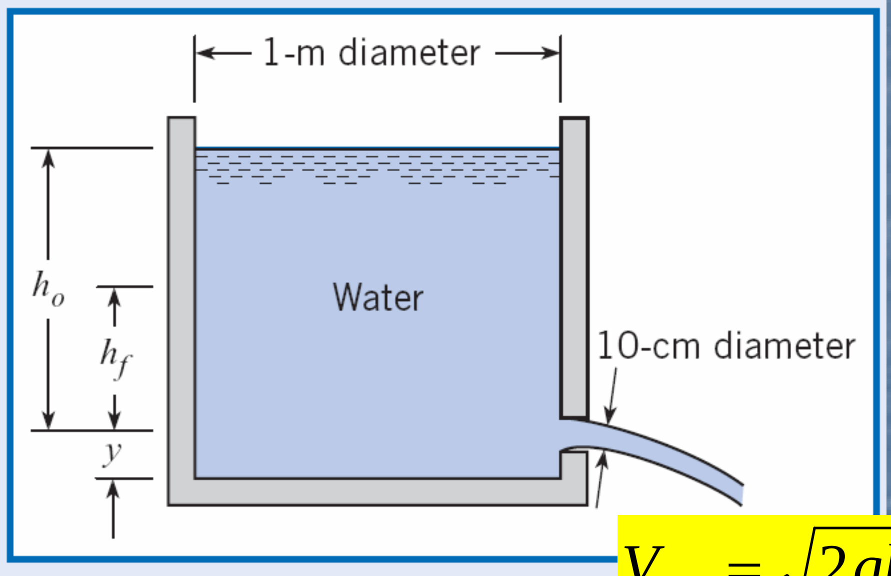
$$-(AV)_{out} = \frac{dh}{dt}(A_T)$$

$$-(A\sqrt{2gh})_{out} = \frac{dh}{dt}(A_T)$$

$$dt = -\frac{A_T}{\sqrt{2g}A_{out}} h^{-1/2} dh$$

$$h_0 = 2m$$

$$h_f = 0.5m$$



$$V_{jet} = \sqrt{2gh}$$

Noting now that $A_T/\sqrt{2g}A_1$ is constant, we integrate the differential equation and get

$$t = \frac{-2A_T}{\sqrt{2g}A_1}h^{1/2} + C$$

The constant of integration is evaluated by arbitrarily letting $t = 0$ when $h = h_0$. Then

$$C = +\frac{2A_T}{\sqrt{2g}A_1}h_0^{1/2}$$

So we have

$$t = \frac{2A_T}{\sqrt{2g}A_1}(h_0^{1/2} - h^{1/2})$$

Thus, for this particular example, the elapsed time for the water level to drop from $h_0 = 2$ m to $h_f = 0.50$ m will be

$$t = \frac{2A_T}{\sqrt{2g}A_1}(2^{1/2} - 0.5^{1/2})$$

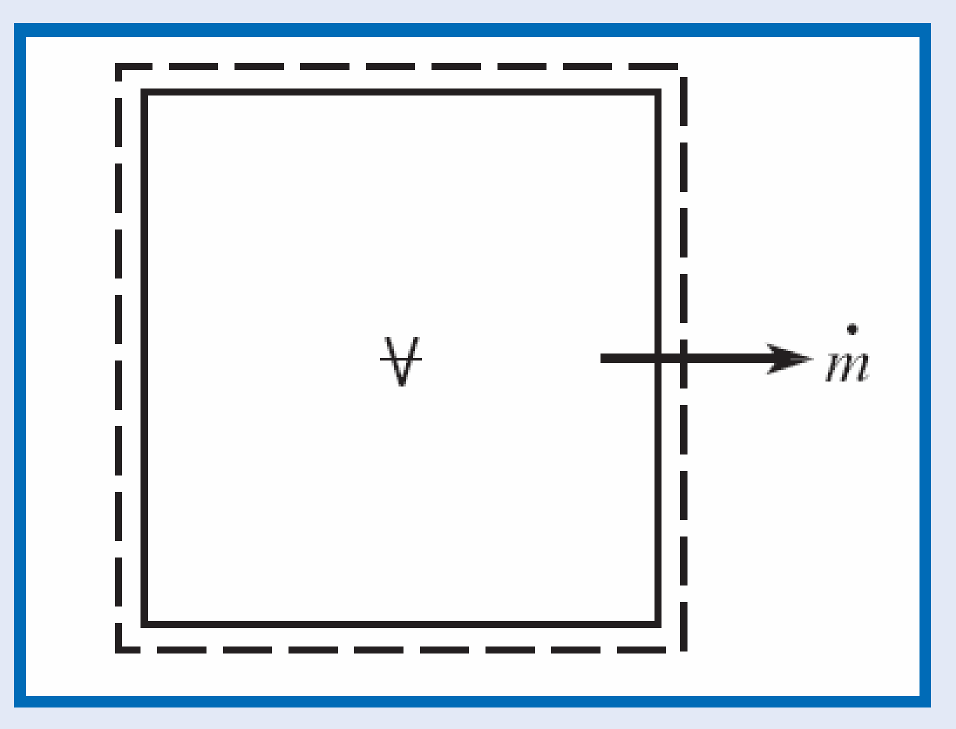
But $$A_T = \frac{\pi}{4}D^2 = \frac{\pi}{4} \times 1^2 = \frac{\pi}{4}\ \text{m}^2$$

$$A_1 = \frac{\pi}{4}(0.10)^2 = 0.01\left(\frac{\pi}{4}\right)\ \text{m}^2$$

Hence $$t = \frac{2\pi/4}{\sqrt{2g}(\pi/4 \times 0.01)}(1.414 - 0.707) = 31.9\ \text{s}$$ ◁

Example 5.8 (p. 159)

Methane escapes through a small (10^{-7} m^2) hole in a 10-m^3 tank. The methane escapes so slowly that the temperature in the tank remains constant at 23°C. The mass flow rate of methane through the hole is given by $\dot{m} = 0.66pA/\sqrt{RT}$, where p is the pressure in the tank, A is the area of the hole, R is the gas constant, and T is the temperature in the tank. Calculate the time required for the absolute pressure in the tank to decrease from 500 to 400 kPa.



Find the time required for the Absolute pressure in the tank to decrease From (500 to 400) kPa ?

Solution Consider a control surface that just encloses the tank. Writing the continuity equation for the control volume, we have

$$\frac{d}{dt}\int_{cv} \rho \, d\forall + \dot{m}_o - \dot{m}_i = 0$$

$$\frac{d}{dt}(\rho \forall) + \dot{m} = 0$$

The tank volume is constant, so

$$d\rho / dt = -\dot{m} / \forall$$

$p = \rho RT$ $dp = d\rho(RT)$

$$\frac{dp}{p} = -0.66\frac{A\sqrt{RT}dt}{\forall}$$

$$t = \frac{1.52\forall}{A\sqrt{RT}}\ln\frac{p_0}{p}$$

where p_0 is the initial pressure. Substituting in the appropriate values, we calculate

$$t = \frac{1.52(10\ \text{m}^3)}{(10^{-7}\ \text{m}^2)\left(518\frac{\text{J}}{\text{kg}\cdot\text{K}}300\ \text{K}\right)^{1/2}}\ln\frac{500}{400} = 8.6\times10^4\ \text{s}$$

◁

or approximately 1 day.

CONTROL VOLUME APPROACH

END OF LECTURE (4)